Lecture 15: More trigonometric substitutions

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Usually, there will be a complex expression in terms of x, and we have to substitute $x = \sin(\theta)$ or $x = 2\tan(\theta)$.

<u>Examples</u>

$$\int \frac{\sqrt{9-x^2}}{x^2}\,\underline{dx}$$

$x = 3\sin(\theta)$

$dx = \underline{3\cos(\theta)\,d\theta}$

$$= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} * \underline{3\cos\theta\,d\theta}$$

$$= \int \frac{\sqrt{9(1-\sin^2\theta)}}{9\sin^2\theta} * 3\cos\theta\,d\theta$$

$$= \underbrace{\cancel{\frac{3}{9} * \sqrt{9}}}_{1} \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta} * \cos\theta\,d\theta$$

$$= \int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta} * cos\theta\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \cot^2\theta\,d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$

$$= -\cot\left(\arcsin\left(\frac{x}{3}\right)\right) - \arcsin\left(\frac{x}{3}\right) + C$$

$$\int_0^{2\pi} \sin^3 x * \cos^4 x\,dx$$

*if one power is odd, use the following identity:

$\sin^2 x + \cos^2 x = 1$

and make the odd power even by splitting it.

ie. $\sin^3 x$ is an odd function, so:

$$\int_0^{2\pi} \sin x * (1-\cos^2 x) * \cos^4 x\,dx$$

$u = \cos x$

$du = -\sin x\,dx$

$$= \int_{\cos(0)}^{\cos(2\pi)} \cancel{\sin x}\,(1-u^2)u^4 * \frac{du}{-\cancel{\sin x}}$$

$$= -\int_1^1 u^4 - u^6\,du$$

same bounds = 0

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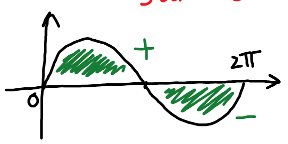


same area

$$= \int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

$\underbrace{\quad}$ we want the structure $\sqrt{1+x^2}$

$x = 2\tan(\theta)$ $\qquad \theta = \arctan\left(\frac{x}{2}\right)$

$dx = 2\sec^2(\theta)\,d\theta$

$$= \int \frac{2\sec^2(\theta)}{(2\cdot\tan(\theta))^2 * \sqrt{(2\tan(\theta))^2+4}}\,d\theta$$

$$= 2\int \frac{\sec^2(\theta)}{4\tan^2\theta * \sqrt{4(\tan^2\theta+1)}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec^2(\theta)}{\tan^2\theta * \sqrt{(\tan^2\theta+1)}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec^{\cancel{2}}(\theta)}{\tan^2\theta * \cancel{\sqrt{\sec^2\theta}}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\frac{1}{\cancel{\cos\theta}}}{\frac{\sin^2\theta}{\cos^{\cancel{2}}\theta}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \frac{1}{4}\int \frac{\cancel{\cos\theta}}{u^2}\frac{du}{\cancel{\cos\theta}}$$

$$= \frac{1}{4}\int u^{-2}\,du$$

$$= \frac{1}{4}\left(\frac{u^{-1}}{-1}\right)$$

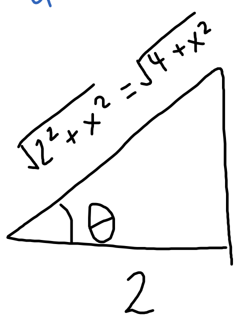
$$= -\frac{1}{4u}$$

we've gone from $x \to \theta \to u$

$$= -\frac{1}{4\sin\theta}$$

$$= -\frac{1}{4\left(\frac{x}{\sqrt{4+x^2}}\right)}$$

$$= -\frac{\sqrt{4+x^2}}{4x} + C$$



$\sin\theta = \frac{x}{\sqrt{4+x^2}}$

$\cos\theta = \frac{2}{\sqrt{4+x^2}}$

ln(c)

2

$$\int_1^3 5^{3x}\,dx$$ $5 = e^{\ln(5)}$

$$= \int_1^3 \left(e^{\ln 5}\right)^{3x} dx$$

$$= \int_1^3 e^{\ln 5 * 3x}\,dx$$

$u = 3x\ln 5$
$du = 3\ln 5\,dx$

$$= \int_{3(1)\ln 5}^{3(3)\ln 5} e^u * \frac{du}{3\ln 5}$$

$$= \int_{3\ln 5}^{9\ln 5} \frac{e^u}{3\ln 5} du = \int_{3\ln 5}^{9\ln 5} \frac{1}{3\ln 5} e^u\,du$$

$$= \frac{1}{3\ln 5}\left(e^{9\ln 5} - e^{3\ln 5}\right) = \frac{1}{3\ln 5}(5^9 - 5^3)$$